

A simple derivation of the electromagnetic field of an arbitrarily moving charge

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Abstract

The expression for the electromagnetic field of a charge moving along an arbitrary trajectory is obtained in a direct, elegant, and Lorentz invariant manner without resorting to more complicated procedures such as differentiation of the Liénard-Wiechert potentials. The derivation uses arguments based on Lorentz invariance and a physically transparent expression originally due to J. J. Thomson for the field of a charge that experiences an impulsive acceleration.

I. INTRODUCTION

Determining the electromagnetic field of a charge moving along an arbitrary trajectory is a standard textbook problem in electrodynamics. Conventionally, the electromagnetic field of a charge moving along an arbitrary trajectory is obtained by solving the wave equation to obtain the Liénard-Wiechert potentials and then calculating the required derivatives of the potentials with respect to position and time.¹ The Liénard-Wiechert potentials are found to be expressible as functions of the retarded time, which is an implicit function of the position at which the field is being evaluated and the trajectory of the charge. This implicit dependence makes differentiating the potentials an elaborate procedure, involving complicated algebra and obscuring the physical significance of the final result.

We present here an alternative derivation of the electromagnetic field of an arbitrarily moving charge, using manifestly Lorentz invariant, four-dimensional notation, without ever obtaining the Liénard-Wiechert potentials or differentiating any expression with respect to the retarded time.

The key idea is that we can derive the electromagnetic fields of a charge in the Lorentz frame at which the charge was instantaneously at rest at the retarded time. In principle, making a Lorentz transformation will give the fields in the original frame.² In practice, the second step turns out to be as laborious as the standard derivation and the algebra is complicated and unilluminating. In addition to the fact that Lorentz transformations mix up \mathbf{E} and \mathbf{B} , we need to express all the original 3-vectors in terms of the corresponding 3-vectors in the new frame. In doing so, we cannot use the standard Lorentz transformation equations which are valid when the velocity is along the x-axis (say), but must instead use the more complicated Lorentz formula for the velocity along an arbitrary direction. Therefore, we adopt a different, manifestly Lorentz invariant approach to tackle the problem, but in the same spirit.

The essential physical ingredients in this derivation are the following: The electromagnetic field at the observation point can depend only on the relative position, the velocity, and the acceleration of the charge, all evaluated at the retarded time, but not on further derivatives of the trajectory. The electromagnetic field of a charge that experiences an impulsive acceleration, that is, an arbitrary but finite acceleration for a short time interval, can be obtained from physical arguments. This derivation is originally due to J. J. Thomson.³ The

generalization of the Thomson formula to a charge moving with any nonzero (and possibly relativistic) velocity is possible by using arguments based on Lorentz invariance.

The plan of the paper is as follows. In Sec. II we construct the most general expression for the electromagnetic field tensor F^{ab} in terms of the variables in the problem. This general expression contains three arbitrary functions of the Lorentz scalars, which need to be determined using further information. In Sec. III we fix these functions by using an independent physical argument based on the Thomson formula and thus determine F^{ab} completely. Our resulting expression for the electric field matches that obtained by the standard procedure of differentiating the Liénard-Wiechert potentials. We summarize our results in a brief concluding section.

II. THE GENERAL EXPRESSION FOR F^{ab} DUE TO A CHARGE

Consider a charge moving along an arbitrary trajectory $z^i(\tau)$ whose electromagnetic field $F^{ab}(x^i)$ at the observation point x^i is to be evaluated. We will use four-dimensional notation with signature $-, +, +, +$; the Latin letters i, j, k, \dots run through $0, 1, 2, 3$ and the Greek letters α, β, γ , denote the spatial components $1, 2, 3$. We shall use units in which $c = 1$. In the conventional approach the current four-vector $J^i(x^i)$ is first obtained from the trajectory of the charge. Then, using the relation $\square A^i = -4\pi J^i$, the four-potential A^i (the Liénard-Wiechert potentials) is calculated. The electromagnetic field tensor F^{ij} is then found by the relation $F^{ij} = \partial^i A^j - \partial^j A^i$. The definition makes clear that F^{ij} is antisymmetric, that is, $F^{ij} = -F^{ji}$. The $F^{0\alpha}$ terms give the components of the electric field, and the $F^{\alpha\beta}$ terms lead to the components of the magnetic field. This procedure of differentiating the Liénard-Wiechert potentials with respect to x^i is tedious because these potentials are expressible only as functions of the retarded proper time τ_{ret} which is an implicit function of x^i and $z^i(\tau)$.

In our alternative approach to this problem we begin with a simple physical idea. We note that the electromagnetic field at the observation point x^i may depend only on the relative position $R^i = x^i - z^i(\tau)$, the velocity u^i , and the acceleration $a^i = du^i/d\tau$ of the charge, all evaluated at the retarded time τ_{ret} , but not on further derivatives of the trajectory. This result arises from the following:

- (a) Because electromagnetic signals propagate at the speed of light, the field at x^i is

determined by the state of the source at an earlier position $z^i(\tau_{\text{ret}})$ which is related to x^i by a null line; that is, by the condition $R_i R^i = 0$. Of the two roots to this equation, we choose the retarded (causal) solution that satisfies the condition $R^0 > 0$. This condition determines the retarded time τ_{ret} .

(b) Translational invariance implies that the field depends only on the relative position R^i of the charge with respect to the observation point (evaluated at the retarded time), and not on the absolute positions of the source or the observation point separately.

(c) Because $\square A^i \sim J^i$, F^{ik} satisfies $\square F^{ik} \sim \partial^i J^k - \partial^k J^i$. Because J^i is at most linear in the velocity of the charge, $\partial^i J^k$ is at most linear in the acceleration, and no further derivatives of the trajectory can occur in the solution F^{ik} .

Therefore, F^{ij} is a second rank antisymmetric tensor which is built from R^i , u^i , and a^i . At this stage it is convenient to introduce the Lorentz invariant scalar $\ell = R_i u^i$ which in the rest frame of the charge reduces to

$$\ell = R_i u^i = -R^0 = -|\mathbf{R}| \equiv -R, \quad (1)$$

where $(R^0)^2 = |\mathbf{R}|^2$ because of the condition $R_i R^i = 0$ and $R^0 > 0$ for the retarded solution. For simplicity, we will also define a four-vector n^i through the relation $R^i \equiv -\ell(n^i + u^i)$. It is easy to see that $n_k u^k = 0$, and $n_k n^k = 1$. The components of n^i are

$$n^i = \left(-\frac{R}{\ell} - \gamma, -\frac{\mathbf{R}}{\ell} - \gamma \mathbf{v} \right), \quad (2)$$

which reduces in the rest frame of the charge to the unit spatial vector pointing from the charge to the field point: $n^i = (0, \mathbf{1})$. We will trade off the R^i dependence of F^{ij} for the n^i dependence and treat F^{ij} as a function of n^i , u^i , and a^i (instead of R^i , u^i , and a^i). Hence the most general form of F^{ij} can be written as

$$F^{ij} = f_1 a^{[i} u^{j]} + f_2 n^{[i} a^{j]} + f_3 n^{[i} u^{j]}, \quad (3)$$

where the square brackets denote antisymmetrization (that is, $a^{[i} u^{j]} = a^i u^j - a^j u^i$) and the functions f_1 , f_2 , f_3 can depend on the scalars that can be constructed from n^i , u^i , and a^i . Because $u_i u^i = -1$, $a_i u^i = 0$, $n_i u^i = 0$, and $n_i n^i = 1$, the only non-trivial scalars which are available to us are $g \equiv n_i a^i$, $R_i u^i = \ell$, and $a_i a^i$. Because F^{ij} is at most linear in a^i , it cannot depend on $a_i a^i$, and hence f_1 , f_2 , and f_3 can depend only on the remaining two scalars g and ℓ . Further, the condition of linearity in a^i also requires that f_1 and f_2 cannot depend

on g (because these functions are multiplied by a term linear in a^i) and can depend only on ℓ . Hence, the most general expression for F^{ij} in terms of n^i , u^i , and a^i becomes:

$$F^{ij} = f_1(\ell)a^{[i}u^{j]} + f_2(\ell)n^{[i}a^{j]} + f_3(g, \ell)n^{[i}u^{j]}, \quad (4)$$

with f_1 , f_2 , and f_3 arbitrary functions of the scalars which are as yet undetermined.

We comment that in constructing the most general second rank antisymmetric tensor out of n^i , u^i , and a^i , we can also include the three terms of the form $\epsilon_{ijkl}a^{[i}u^{j]}$ involving the totally antisymmetric tensor ϵ_{ijkl} . The reason for omitting terms of this type in Eq. (4) is as follows. When ϵ_{ijkl} multiplies an antisymmetric (true) tensor A^{ij} , for example, we obtain the dual tensor $*A^{ij}$ which is not a true tensor, but a pseudotensor; that is, it transforms differently from true tensors under reflections of the co-ordinate system. We know that the electromagnetic field tensor F^{ij} is a true tensor. Hence, we cannot have pseudotensorial terms such as $\epsilon_{ijkl}a^{[i}u^{j]}$ in the general expression for F^{ij} .

Our next task is to determine f_1 , f_2 , and f_3 , which will completely solve the problem. We proceed by considering the two four vectors E^i and B^i defined as

$$E^i = u_j F^{ij}, \quad B^i = \frac{1}{2} \epsilon^{ijkl} u_j F_{kl}. \quad (5)$$

The vectors E^i and B^i contain the same amount of information as F^{ij} as can be seen by the explicit expression for the latter in terms of the former:

$$F^{ij} = u^i E^j - E^i u^j - \epsilon^{ij}_{\quad kl} u^k B^l, \quad (6)$$

which can be easily verified by direct substitution of Eq. (6) into Eq. (5) and the use of the identities $u_j E^j = 0$ and $u_j B^j = 0$. Incidentally, these identities, which follow from the definitions in Eq. (5) and the antisymmetry of F^{ij} and ϵ^{ijkl} , also show that E^i and B^i are both orthogonal to u^i , and hence, in a given reference frame, they contain only three independent components. We substitute Eq. (4) into Eq. (5) and find

$$E^i = -f_1(\ell)a^i - f_3(g, \ell)n^i, \quad B^i = f_2(\ell)\epsilon^{ijkl}u_j n_k a_l. \quad (7)$$

If we know the explicit forms of E^i and B^i in terms of the variables in the problem, we can determine the functions f_1 , f_2 , and f_3 by direct comparison.

The four vectors E^i and B^i have direct physical interpretations and represent the electric and magnetic fields in the instantaneous rest frame of the charge with four velocity u^i . In

the instantaneous rest frame of the charge in which $u^j = (1, \mathbf{0})$ only the component u_0 contributes and $E^i = u_0 F^{i0} = (0, F^{0\alpha})$, because F^{ij} is antisymmetric. Hence, the spatial components of $E^i = u_j F^{ij}$ correctly represent the components of the electric field in the instantaneous rest frame of the charge. Similarly, in the instantaneous rest frame, only the component u_0 contributes to B^i . Because ϵ^{ijkl} is completely antisymmetric, the time component of B^i vanishes in this frame. We see that the spatial components of B^i are given by $F^{\alpha\beta}$ where $\alpha, \beta = 1, 2$, or 3 . Hence the spatial components of B^i lead to the correct values of the magnetic field components in the rest frame.

We now will obtain explicit expressions for the (three-dimensional) electric and magnetic field vectors in the rest frame by using an independent physical argument. From these explicit expressions we will construct two four-vectors, \mathcal{E}^i and \mathcal{B}^i , such that the spatial components of \mathcal{E}^i and \mathcal{B}^i are equivalent to the electric and magnetic three-vectors in the rest frame and the time components of \mathcal{E}^i and \mathcal{B}^i vanish in this frame. If we use the fact that \mathcal{E}^i and \mathcal{B}^i are generally covariant four-dimensional quantities and are identical in the rest frame with the original E^i and B^i (by construction), we can equate the general expressions, E^i to \mathcal{E}^i and B^i to \mathcal{B}^i , and thereby determine the functions f_1 , f_2 , and f_3 .

III. ELECTROMAGNETIC FIELDS IN THE INSTANTANEOUS REST FRAME OF THE CHARGE

The physical argument to determine the electric and magnetic field vectors is based on the physically transparent expression for the electromagnetic field due to J. J. Thomson.³ This expression is usually given for a charged particle moving with non-relativistic velocities and becomes exact in the limit of $v \rightarrow 0$. Its derivation is reproduced and discussed in standard texts.^{4,5} For convenience we give a brief derivation in the Appendix. The analysis considers a charge that is moving with a uniform (non-relativistic) velocity, experiences a finite acceleration \mathbf{a} for an infinitesimal time interval Δt , and then continues to move with uniform velocity. The expression for the electromagnetic field under these conditions can be expressed as:

$$\mathbf{E} = \mathbf{E}_{\text{coul}} + \mathbf{E}_{\text{rad}}, \quad \mathbf{B} = \hat{\mathbf{n}} \times \mathbf{E}_{\text{rad}}, \quad (8)$$

where

$$\mathbf{E}_{\text{coul}} = \frac{q}{R^2} \hat{\mathbf{n}} \quad (9)$$

is the Coulomb field of a charge at rest a distance R from the observation point with $\hat{\mathbf{n}} = \mathbf{R}/R$ the unit vector in the radial direction. Also

$$\mathbf{E}_{\text{rad}} = \frac{q}{R} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{a}) \quad (10)$$

is the radiation field of the charge, transverse to the radial direction, due to the acceleration \mathbf{a} of the charge. We can thus rewrite Eq. (8) as:

$$\mathbf{E} = \frac{q}{R^2} \hat{\mathbf{n}} + \frac{q}{R} [\hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{a}) - \mathbf{a}] \quad (11)$$

$$\mathbf{B} = -\frac{q}{R} [\hat{\mathbf{n}} \times \mathbf{a}]. \quad (12)$$

In conventional units the radiation field terms (in Eqs. (10), (11), and (12)) will each contain a c^2 factor in the denominator.

We now show that the electric and magnetic fields in Eqs. (11) and (12) also represent the electromagnetic field of a charge in arbitrary motion, in the instantaneous rest frame at the retarded time. Although this fact is intuitively obvious, we give a formal argument to make our derivation complete. Let us compare the fields of two charges C and C_1 that have two different trajectories. The charge C is the one of interest and moves along an arbitrary trajectory. We make a Lorentz transformation to the instantaneous rest frame (at the retarded time) of C . In this frame the charge C is at rest, but it has an instantaneous acceleration \mathbf{a} . The charge C_1 is chosen to have the same acceleration and position as charge C for a small time interval Δt around the retarded time and moves with a uniform velocity later on. The position (at the origin), velocity (equal to zero), and acceleration (equal to \mathbf{a}) of C and C_1 are identical at the retarded time, and because the electromagnetic field depends only on these quantities, both charges will lead to the same fields at the observation point. The trajectories of the two charges are quite different after a time Δt (with C moving along some arbitrary curve while C_1 moves with a uniform velocity), but the expression for the radiation field, Eq. (10), cannot depend on the future trajectory of charge C . Therefore, we have obtained the important result that in the instantaneous rest frame of the charge C , Eqs. (11) and (12) give the correct form of the electromagnetic field.

Equations (11) and (12) are three-dimensional expressions for the electric and magnetic fields. We use these expressions to now construct four-vectors, \mathcal{E}^i and \mathcal{B}^i , such that their spatial components give the above three-vectors in the rest frame, and their time components vanish in that frame. We begin by noting that R in Eq. (10) is the value of the Lorentz scalar

$-R_i u^i = -\ell$ in the rest frame. Also, in the rest frame we have $n^i = (0, \mathbf{1})$ and $a_i = (0, \mathbf{a})$. We use these results to generalize Eq. (11) to four-dimensions as

$$\mathcal{E}^i = \frac{q}{\ell^2} n^i + \frac{q}{\ell} [a^i - n^i (n_k a^k)]. \quad (13)$$

The structure of this four vector is unique and easy to understand. We would have expected it to be a linear combination of a^i , n^i , and u^i with coefficients depending on the scalars built out of them. But a term linear in u^i cannot occur because u^i in the rest frame has only the time component but a^i and n^i have only space components. So their linear combination cannot vanish in the rest frame (and give $\mathcal{E}^0 = 0$) if there is a term with u^i . The coefficients of a^i and n^i are uniquely fixed by the form of Eq. (11). Also, note that \mathcal{E}^i contains one term which goes as $1/\ell^2$ and one term which goes as $1/\ell$, which come from the three-dimensional Coulomb field and radiation field respectively.

To generalize Eq. (12) we note the appearance of the cross product. In three dimensions a cross product is represented by making use of the completely antisymmetric tensor $\epsilon^{\alpha\beta\gamma}$. We thus need a tensor that goes over to $\epsilon^{\alpha\beta\gamma}$ in the instantaneous rest frame. Consider the tensor $u_i \epsilon^{ijkl}$. In the rest frame, as mentioned before, the contribution is only from u_0 . Because ϵ^{ijkl} is antisymmetric, all the components of $u_i \epsilon^{ijkl}$ when j , k , or l is zero, vanish in the rest frame. Hence, the non-zero components of the tensor $u_i \epsilon^{ijkl}$ are identical to those of $-\epsilon^{\alpha\beta\gamma}$ in the rest frame. Hence, $u_i \epsilon^{ijkl}$ is the tensor we need and Eq. (12) can thus be generalized as

$$\mathcal{B}^i = \frac{q}{\ell} \epsilon^{ijkl} u_j n_k a_l. \quad (14)$$

We see that in the rest frame, the time component of Eq. (14) vanishes and the spatial components give the magnetic field.

All that remains is to equate the explicit forms for \mathcal{E}^i and \mathcal{B}^i in Eqs. (13) and (14) to the general expressions for E^i and B^i in Eq. (7) involving the undetermined functions f_1 , f_2 , and f_3 and determine the latter. The equality $\mathcal{E}^i = E^i$ gives

$$-f_1(\ell) a^i - f_3(g, \ell) n^i = \frac{q}{\ell^2} n^i + \frac{q}{\ell} a^i - \frac{q}{\ell} n^i (n_k a^k). \quad (15)$$

Equation (15) determines the functions f_1 and f_3 to be (recall that $g = n_k a^k$)

$$f_1(\ell) = -\frac{q}{\ell}, \quad f_3(g, \ell) = \frac{q}{\ell} (n_k a^k) - \frac{q}{\ell^2}. \quad (16)$$

Similarly, the equality $\mathcal{B}^i = B^i$ gives

$$f_2(\ell)\epsilon^{ijkl}u_jn_ka_l = \frac{q}{\ell}\epsilon^{ijkl}u_jn_ka_l, \quad (17)$$

and fixes f_2 to be

$$f_2(\ell) = \frac{q}{\ell}. \quad (18)$$

Thus we obtain the final expression for F^{ij} :

$$F^{ij} = \frac{q}{\ell^2}u^{[i}n^{j]} + \frac{q}{\ell}[n^{[i}a^{j]} - a^{[i}u^{j]} + (n_ka^k)n^{[i}u^{j]}]. \quad (19)$$

Equation (19) completely solves the problem. The electromagnetic field tensor F^{ij} naturally splits into a ‘‘Coulomb’’ term and a ‘‘radiation’’ term.

As an aside, we note that there is an alternative and somewhat shorter way of arriving at Eq. (19). We substitute the four-dimensional generalized fields derived from the Thomson expression, namely \mathcal{E}^i and \mathcal{B}^i , for E^i and B^i respectively into Eq. (6) and obtain the explicit expression for F^{ij} without first arriving at the expression in Eq. (4). If we substitute Eqs. (13) and (14) into Eq. (6), we obtain

$$F^{ij} = \frac{q}{\ell^2}u^{[i}n^{j]} - \frac{q}{\ell}a^{[i}u^{j]} + \frac{q}{\ell}(n_ka^k)n^{[i}u^{j]} - \frac{q}{\ell}\epsilon^{ij}{}_{kl}\epsilon^{lpqr}u^kn_qa_ru_p. \quad (20)$$

To evaluate the expression $\epsilon^{ij}{}_{kl}\epsilon^{lpqr}u^kn_qa_ru_p$ we use the identity

$$\epsilon_{ijkl}\epsilon^{lpqr} = -[\delta_i^p(\delta_j^r\delta_k^q - \delta_j^q\delta_k^r) - \delta_j^r(\delta_i^p\delta_k^q - \delta_k^p\delta_i^q) + \delta_i^q(\delta_j^p\delta_k^r - \delta_j^r\delta_k^p)]. \quad (21)$$

We lower the indices i and j in Eq. (20) and then use Eq. (21) to obtain the expression for F_{ij} :

$$F_{ij} = \frac{q}{\ell^2}u_{[i}n_{j]} - \frac{q}{\ell}a_{[i}u_{j]} + \frac{q}{\ell}(n_ka^k)n_{[i}u_{j]} + \frac{q}{\ell}n_{[i}a_{j]}. \quad (22)$$

Raising the indices i and j reproduces Eq. (19).

The final result, Eq. (19), is known in the literature. It is obtained by integrating Maxwell’s equations in a four-dimensional notation, and differentiating the resultant Liénard-Wiechert potentials A^j with respect to x^i .^{4,6}

As a direct check of Eq. (19), we evaluate the $F^{0\alpha}$ components of Eq. (19). To do so we expand Eq. (19) with $i = 0$ and $j = \alpha$ in terms of the zero and α components of the

four-vectors u^i , n^i , and a^i . If we use

$$n^i = \left(-\frac{R}{\ell} - \gamma, -\frac{\mathbf{R}}{\ell} - \gamma\mathbf{v} \right) \quad (23)$$

$$u^i = (\gamma, \gamma\mathbf{v}) \quad (24)$$

$$a^i = (\gamma^4\mathbf{v} \cdot \mathbf{a}, \gamma^2\mathbf{a} + \gamma^4(\mathbf{v} \cdot \mathbf{a})\mathbf{v}) \quad (25)$$

(obtained by calculating $du^i/d\tau$), and the fact that in the lab frame, the value of ℓ is $R_i u^i = \gamma(-R + \mathbf{v} \cdot \mathbf{R})$, it is easy to show that

$$F^{0\alpha} = \frac{q(1-v^2)}{(R - \mathbf{v} \cdot \mathbf{R})^3}(\mathbf{R} - \mathbf{v}R) + \frac{q}{(R - \mathbf{v} \cdot \mathbf{R})^3}(\mathbf{R} \times ((\mathbf{R} - \mathbf{v}) \times \mathbf{a})), \quad (26)$$

which is identical to the expression for the electric field obtained by differentiating the standard Liénard-Wiechert potentials.¹

IV. CONCLUSION

We have derived the general expression for the electromagnetic field of an arbitrarily moving charge in a comparatively simple manner. This approach fully exploits the nature of the dependence of the electromagnetic field tensor on the dynamical variables. The only other ingredient used is the Thomson expression, which is physically transparent and fairly straightforward to derive. The algebra is simplified considerably by the use of simple and intuitive physical arguments. Our derivation, apart from pedagogical interest, also yields the explicit expression for the electromagnetic field tensor F^{ij} in terms of the position, velocity, and acceleration four-vectors. The use of four-dimensional variables and Lorentz invariant quantities does away with the need to perform any Lorentz transformations or differentiate complicated implicit functions with respect to the retarded time.

APPENDIX: DERIVATION OF THE THOMSON FORMULA

We briefly derive the Thomson formula given in Eq. (8). Consider a charged particle which is at rest until time $t = 0$, experiences an acceleration \mathbf{a} for an infinitesimal time interval Δt , and then continues to move with uniform velocity. The electric field lines of the charge as observed at a time $t \gg \Delta t$ are shown in Fig. 1(a). We are interested in

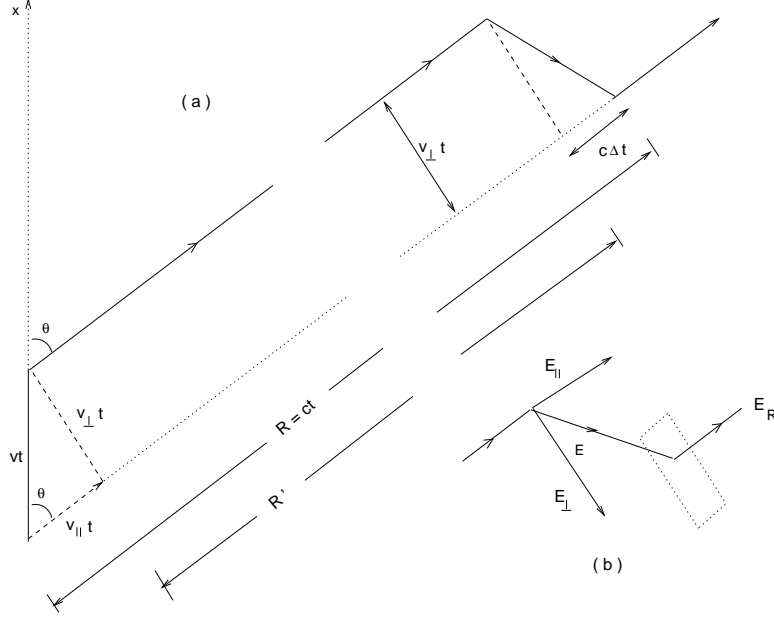


FIG. 1: (a) Electric field lines of a charge accelerated for an infinitesimal time interval Δt . (b) Gaussian pillbox used to relate E_{\parallel} to E_R .

the limit $\Delta t \rightarrow 0$ with \mathbf{a} finite such that the final velocity of the charge is nonrelativistic, $v = a\Delta t \ll c$. In this Appendix we have re-introduced the c -factor.

At distances $R > ct$ the field is that of a charge at rest: $E_R = q\hat{\mathbf{n}}/R^2$. At distances $R' < c(t - \Delta t)$ the field is that of a charge moving with uniform velocity, which for non-relativistic velocities can be approximated to that of a charge at rest: $E_{R'} = q\hat{\mathbf{n}}'/R'^2$. Between these two limits the acceleration \mathbf{a} of the charge produces kinks in the field lines and leads to the production of a transverse radiation field E_{rad} in addition to the radial Coulomb field. To find the magnitude of E_{rad} we resolve the velocity \mathbf{v} , acceleration \mathbf{a} , and total electric field \mathbf{E} into components perpendicular and parallel to the radial unit vector $\hat{\mathbf{n}}$. Similarity of triangles then gives

$$\frac{E_{\perp}}{E_{\parallel}} = \frac{v_{\perp}t}{c\Delta t}. \quad (\text{A.1})$$

By using $v_{\perp} = a_{\perp}\Delta t$ and $t = R/c$, we get

$$\frac{E_{\perp}}{E_{\parallel}} = \frac{a_{\perp}R}{c^2}. \quad (\text{A.2})$$

Electric flux conservation applied to the Gaussian pillbox shown in Fig. 1(b), equates E_{\parallel}

with $E_R = q/R^2$ and thus leads to

$$\mathbf{E}_\perp = \frac{-q\mathbf{a}_\perp}{c^2 R}, \quad (\text{A.3})$$

which, on noting that $\mathbf{a}_\perp = \mathbf{a} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{a})$, readily gives Eq. (10).

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